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Matrix Recursive Projection and Interpolation Algorithms

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ABSTRACT

The Schur complement is used with the block bordering method to derive two recursive algorithms called the matrix recursive projection algorithm and the matrix recursive interpolation algorithm. These algorithms are extensions of the RPA and the RIA studied by Brezinski in the vector case. Some properties of these algorithms are given.

1. INTRODUCTION

Let E be a vector space over a field K, and E^* its dual. We denote by $\langle \cdot, \cdot \rangle$ the bilinear form of the duality between E and E^*. Let $y \in E$, $x_i \in E$ and $z_i \in E^*$. We set

$$n_k = \begin{vmatrix} y & x_1 & \cdots & x_k \\ \langle z_1, y \rangle & \langle z_1, x_1 \rangle & \cdots & \langle z_1, x_k \rangle \\ \vdots & \vdots & & \vdots \\ \langle z_k, y \rangle & \langle z_k, x_1 \rangle & \cdots & \langle z_k, x_k \rangle \end{vmatrix},$$

$$d_k = \begin{vmatrix} \langle z_1, x_1 \rangle & \cdots & \langle z_1, x_k \rangle \\ \vdots & & \vdots \\ \langle z_k, x_1 \rangle & \cdots & \langle z_k, x_k \rangle \end{vmatrix},$$

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$$n_{k,i} = \begin{vmatrix} x_i & x_1 & \cdots & x_k \\ \langle z_1, x_i \rangle & \langle z_1, x_1 \rangle & \cdots & \langle z_1, x_k \rangle \\ \vdots & \vdots & & \vdots \\ \langle z_k, x_i \rangle & \langle z_k, x_1 \rangle & \cdots & \langle z_k, x_k \rangle \end{vmatrix}.$$

The generalized determinants n_k and $n_{k,i}$ are the elements of E obtained by expansion with respect to their first row by using the classical rule for expanding a determinant. We set

$$e_k = \frac{n_k}{d_k}, \qquad g_{k,i} = \frac{n_{k,i}}{d_k}, \quad \text{and} \quad p_k = y - e_k,$$

and we assume that for $m = 1, \ldots, k$, $d_m \neq 0$. It has been proved in [1] that the e_k's and p_k's can be recursively computed by the following algorithms:

(1) the *recursive projection algorithm* (RPA):

$$e_0 = y, \qquad g_{0,i} = x_i \quad \text{for } i = 1, \ldots, k,$$

and for $m = 1, \ldots, k$,

$$e_m = e_{m-1} - \frac{\langle z_m, e_{m-1} \rangle}{\langle z_m, g_{m-1,m} \rangle} g_{m-1,m},$$

$$g_{m,i} = g_{m-1,i} - \frac{\langle z_m, g_{m-1,i} \rangle}{\langle z_m, g_{m-1,m} \rangle} g_{m-1,m}$$

for $i = m + 1, \ldots, k$; and

(2) the *recursive interpolation algorithm* (RIA):

$$p_0 = 0, \qquad g_{0,i} = x_i \quad \text{for } i = 1, \ldots, k,$$

and for $m = 1, \ldots, k$,

$$p_m = p_{m-1} + \frac{\langle z_m, y\rangle - \langle z_m, p_{m-1}\rangle}{\langle z_m, g_{m-1,m}\rangle} g_{m-1,m},$$

$$g_{m,i} = g_{m-1,i} - \frac{\langle z_m, g_{m-1,i}\rangle}{\langle z_m, g_{m-1,m}\rangle} g_{m-1,m}$$

for $i = m+1, \ldots, k$.

Applications of the RPA and the RIA and their connections with some other methods used in various areas of numerical analysis have been studied in [3, 4]. Our interest here is to study the extension of the RPA and the RIA to the matrix case. In Section 2 we will introduce the Schur complement with the block bordering method (see [2, 5]) for giving, in Section 3, the extension of the RPA and the RIA to the matrix case; in Section 4 we will give some properties of this extension.

2. THE SCHUR COMPLEMENT AND THE BLOCK BORDERING METHOD

2.1. *The Schur Complement*

Let A be a matrix partitioned into four blocks

$$A = \begin{pmatrix} B & U \\ V & M \end{pmatrix}, \tag{2.1}$$

where the matrix M is assumed to be square and nonsingular. The Schur complement of M in A, denoted by (A/M), is defined by

$$(A/M) = B - UM^{-1}V. \tag{2.2}$$

When A is square we have

$$|(A/M)| = \frac{|A|}{|M|}. \tag{2.3}$$

We can similarly define the Schur complements

$$(A/U) = V - MU^{-1}B \qquad \text{if } U \text{ is a nonsingular matrix,} \tag{2.4}$$

$$(A/V) = U - BV^{-1}M \qquad \text{if } V \text{ is a nonsingular matrix,} \tag{2.5}$$

$$(A/B) = M - VB^{-1}U \qquad \text{if } B \text{ is a nonsingular matrix.} \tag{2.6}$$

2.2. The Block Bordering Method

Let B be an $n \times n$ matrix. We consider the $(n+m) \times (n+m)$ matrix A given by

$$A = \begin{bmatrix} B & U \\ V & M \end{bmatrix}, \tag{2.7}$$

where U, V, and M are matrices of the respective dimensions $n \times m$, $m \times n$, and $m \times m$. B and $(A/B) = M - VB^{-1}U$ are assumed to be regular. It is easy to check that

$$A^{-1} = \begin{bmatrix} B^{-1} + B^{-1}U(M - VB^{-1}U)^{-1}VB^{-1} & -B^{-1}U(M - VB^{-1}U)^{-1} \\ -(M - VB^{-1}U)^{-1}VB^{-1} & (M - VB^{-1}U)^{-1} \end{bmatrix}. \tag{2.8}$$

Let us now consider the systems of linear equations

$$Bz = d \quad \text{and} \quad Ay = \begin{bmatrix} d \\ f \end{bmatrix},$$

where f is a vector of dimension m. From the preceding formula for A^{-1} we have

$$y = \begin{bmatrix} z \\ 0 \end{bmatrix} + \begin{bmatrix} -B^{-1}U \\ I \end{bmatrix}(M - VB^{-1}U)^{-1}(f - Vz), \tag{2.9}$$

where I is the $m \times m$ identity matrix. It is proved in [5] that A is singular if and only if $M - VB^{-1}U$ is singular.

We shall apply this block bordering method to the Schur complements for obtaining the extension of the RPA and the RIA to the matrix case.

3. THE MRPA AND THE MRIA

Let Y be an $n \times p$ complex matrix, and for $i = 1, \dots, k$, let X_i and Z_i be the $n \times n_i$ complex matrices, with $n_1 + n_2 + \cdots + n_k \leqslant n$. We consider

$$N_k = \begin{bmatrix} Y & X_1 & \cdots & X_k \\ Z_1^* Y & Z_1^* X_1 & \cdots & Z_1^* X_k \\ \vdots & \vdots & & \vdots \\ Z_k^* Y & Z_k^* X_1 & \cdots & Z_k^* X_k \end{bmatrix}, \tag{3.1}$$

$$D_k = \begin{bmatrix} Z_1^* X_1 & \cdots & Z_1^* X_k \\ \vdots & & \vdots \\ Z_k^* X_1 & \cdots & Z_k^* X_k \end{bmatrix}, \tag{3.2}$$

$$N_{k,i} = \begin{bmatrix} X_i & X_1 & \cdots & X_k \\ Z_1^* X_i & Z_1^* X_1 & \cdots & Z_1^* X_k \\ \vdots & \vdots & & \vdots \\ Z_k^* X_i & Z_k^* X_1 & \cdots & Z_k^* X_k \end{bmatrix}, \tag{3.3}$$

where Z_i^* is the adjoint matrix of Z_i. If $|D_k| \neq 0$, we set

$$E_k = (N_k/D_k), \qquad G_{k,i} = (N_{k,i}/D_k) \tag{3.4}$$

$$P_k = Y - E_k. \tag{3.5}$$

We shall now give two algorithms for computing the E_k's and the P_k's. We have from (2.2)

$$E_k = Y - XD_k^{-1}U. \tag{3.6}$$

where

$$X = [X_1, \dots, X_k] \quad \text{and} \quad U = \begin{bmatrix} Z_1^* Y \\ \vdots \\ Z_k^* Y \end{bmatrix}.$$

The product $D_k^{-1}U$ in (3.6) can be obtained by applying the block bordering method [5]. We set

$$D_k = \begin{bmatrix} D_{k-1} & U'' \\ V & A_{k,k} \end{bmatrix}, \qquad U = \begin{bmatrix} U' \\ U_k \end{bmatrix},$$

where

$$D_{k-1} = \begin{bmatrix} Z_1^* X_1 & \cdots & Z_1^* X_{k-1} \\ \vdots & & \vdots \\ Z_{k-1}^* X_1 & \cdots & Z_{k-1}^* X_{k-1} \end{bmatrix}, \qquad U'' = \begin{bmatrix} Z_1^* X_k \\ \vdots \\ Z_{k-1}^* X_k \end{bmatrix},$$

$$U' = \begin{bmatrix} Z_1^* Y \\ \vdots \\ Z_{k-1}^* Y \end{bmatrix}$$

$$V = [Z_k^* X_1, \ldots, Z_k^* X_{k-1}], \qquad A_{k,k} = Z_k^* X_k, \; U_k = Z_k^* Y.$$

If D_{k-1} and $(D_k/D_{k-1}) = A_{k,k} - VD_{k-1}^{-1}U''$ are nonsingular matrices, we obtain from (2.8)

$$D_k^{-1} = \begin{bmatrix} D_{k-1}^{-1} + D_{k-1}^{-1}U''(D_k/D_{k-1})^{-1}VD_{k-1}^{-1} & -D_{k-1}^{-1}U''(D_k/D_{k-1})^{-1} \\ -(D_k/D_{k-1})^{-1}VD_{k-1}^{-1} & (D_k/D_{k-1})^{-1} \end{bmatrix} \tag{3.7}$$

and from (2.9)

$$D_k^{-1}U = \begin{bmatrix} D_{k-1}^{-1}U' \\ 0 \end{bmatrix} + \begin{bmatrix} -D_{k-1}^{-1}U'' \\ I \end{bmatrix}(D_k/D_{k-1})^{-1}(U_k - VD_{k-1}^{-1}U'). \tag{3.8}$$

If we set $X = [X', X_k]$ with $X' = [X_1, \ldots, X_{k-1}]$, we obtain

$$XD_k^{-1}U = X'D_{k-1}^{-1} + (X_k - X'D_{k-1}^{-1}U'')(D_k/D_{k-1})^{-1}(U_k - VD_{k-1}^{-1}U') \tag{3.9}$$

and

$$E_k = Y - X'D_{k-1}^{-1}U' - \left(X_k - X'D_{k-1}^{-1}U''\right)\left(D_k/D_{k-1}\right)^{-1}$$
$$\times\left(U_k - VD_{k-1}^{-1}U'\right). \tag{3.10}$$

Using (2.2), we have

$$Y - X'D_{k-1}^{-1}U' = \left(\begin{bmatrix} Y & X' \\ U' & D_{k-1} \end{bmatrix}\Big/ D_{k-1}\right) = \left(N_{k-1}/D_{k-1}\right) = E_{k-1},$$

$$X_k - X'D_{k-1}^{-1}U'' = \left(\begin{bmatrix} X_k & X' \\ U'' & D_{k-1} \end{bmatrix}\Big/ D_{k-1}\right) = \left(N_{k-1,k}/D_{k-1}\right) = G_{k-1,k}.$$

We also have

$$\begin{aligned} U_k - VD_{k-1}^{-1}U' &= Z_k^* Y - Z_k^*[X_1, \dots, X_{k-1}]D_{k-1}^{-1}U' \\ &= Z_k^*\left(Y - X'D_{k-1}^{-1}U'\right) \\ &= Z_k^* E_{k-1}, \\ \left(D_k/D_{k-1}\right) &= A_{k,k} - VD_{k-1}^{-1}U'' \\ &= Z_k^*\left(X_k - X'D_{k-1}^{-1}U''\right) \\ &= Z_k^* G_{k-1,k}, \end{aligned}$$

and so we get

$$E_k = E_{k-1} - G_{k-1,k}\left(Z_k^* G_{k-1,k}\right)^{-1} Z_k^* E_{k-1}. \tag{3.11}$$

Since the expression for $G_{k,i}$ is obtained from that of E_k (3.11) by replacing Y with X_i, a similar recursive relation holds for $G_{k,i}$. Thus we finally obtain the following recursive algorithm, called the *matrix recursive projection algorithm* (MRPA):

$$E_0 = Y, \qquad G_{0,i} = X_i \quad \text{for } i = 1, \dots, k,$$

and for $m = 1, \ldots, k$

$$E_m = E_{m-1} - G_{m-1,m}(Z_m^* G_{m-1,m})^{-1} Z_m^* E_{m-1},$$

$$G_{m,i} = G_{m-1,i} - G_{m-1,m}(Z_m^* G_{m-1,m})^{-1} Z_m^* G_{m-1,i}$$

for $i = m+1, \ldots, k$.

We have

$$P_k = Y - E_k,$$

and so, from the MRPA, we obtain the following algorithm, called the *matrix recursive interpolation algorithm* (MRIA):

$$P_0 = 0, \qquad G_{0,i} = X_i \quad \text{for } i = 1, \ldots, k,$$

and for $m = 1, \ldots, k$

$$P_m = P_{m-1} + G_{m-1,m}(Z_m^* G_{m-1,m})^{-1} Z_m^* (Y - P_{m-1}),$$

$$G_{m,i} = G_{m-1,i} - G_{m-1,m}(Z_m^* G_{m-1,m})^{-1} Z_m^* G_{m-1,i}$$

for $i = m+1, \ldots, k$.

REMARK 1.

(1) The MRPA and the MRIA are well defined when $|D_m| \neq 0$ for $m = 1, \ldots, k$.

(2) If we set $Z = [Z_1, \ldots, Z_k]$ we have

$$\begin{aligned} E_k &= Y - X(Z^*X)^{-1} Z^*Y, \\ P_k &= X(Z^*X)^{-1} Z^*Y. \end{aligned} \tag{3.12}$$

(3) When $p = 1$ and $n_i = 1$ for $i = 1, \ldots, k$, we obtain the recursive projection algorithm and the recursive interpolation algorithm studied by Brezinski [3] in the vector case.

(4) When $p = 1$ and $n_i > 1$ for some i, we call the MRPA the *block recursive projection algorithm* (BRPA) and the MRIA the *block recursive interpolation algorithm* (BRIA).

EXAMPLE. Let

$$Y = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let us consider the Schur complement given by (3.4) and the Schur complement given by (3.5) for $k = 2$. Applying the MRPA and the MRIA, we obtain

$$E_0 = Y, \qquad G_{0,1} = X_1, \qquad G_{0,2} = X_2, \qquad P_0 = 0,$$

$$Z_1^* G_{0,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix},$$

$$(Z_1^* G_{0,1})^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix},$$

$$G_{0,1}(Z_1^* G_{0,1})^{-1} Z_1^* = \frac{1}{5} \begin{pmatrix} 4 & 1 & 3 \\ 1 & 4 & -3 \\ 1 & -1 & 2 \end{pmatrix},$$

$$E_1 = E_0 - G_{0,1}(Z_1^* G_{0,1})^{-1} Z_1^* E_0 = \frac{1}{5} \begin{pmatrix} -4 & -2 \\ 4 & 2 \\ 4 & 2 \end{pmatrix},$$

$$P_1 = P_0 + G_{0,1}(Z_1^* G_{0,1})^{-1} Z_1^* (Y - P_0) = \frac{1}{5} \begin{pmatrix} 9 & 2 \\ 6 & -7 \\ 1 & 3 \end{pmatrix},$$

$$G_{1,2} = G_{0,2} - G_{0,1}(Z_1^* G_{0,1})^{-1} Z_1^* G_{0,2} = \frac{1}{5} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

$$G_{1,2}(Z_2^* G_{1,2})^{-1} Z_2^* = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$E_2 = E_1 - G_{1,2}(Z_2^* G_{1,2})^{-1} Z_2^* E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_2 = P_1 + G_{1,2}(Z_2^* G_{1,2})^{-1} Z_2^* (Y - P_1) = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

We remark that $E_2 = 0$ and $P_2 = Y$. We will show in the next section that when $n_1 + n_2 + \cdots + n_k = n$ we have $E_k = 0$ and $P_k = Y$.

4. PROPERTIES

In this section we shall use some properties of projectors (see [6, 7]) for giving some properties of the MRPA and the MRIA. We will show also that when $n_1 + n_2 + \cdots + n_k = n$, we get $E_k = 0$ and $P_k = Y$. Let D_k be the matrix defined by (3.2). We set

$$D_{i,j} = Z_i^* X_j$$

and for $m = 1, \ldots, k$

$$D_m = (D_{i,j})_{1 \leqslant i,j \leqslant m}.$$

DEFINITION. D_k is called a *block strongly nonsingular matrix* if $|D_m| \neq 0$ for $m = 1, \ldots, k$.

LEMMA 4.1. *For* $m = 1, \ldots, k$, *we have*

$$|Z_m^* G_{m-1,m}| = \frac{|D_m|}{|D_{m-1}|} \qquad \textit{with} \quad |D_0| = 1.$$

Proof. We have

$$G_{m-1,m} = \left(\begin{bmatrix} X_m & X_1 & \cdots & X_{m-1} \\ Z_1^* X_m & Z_1^* X_1 & \cdots & Z_1^* X_{m-1} \\ \vdots & \vdots & & \vdots \\ Z_{m-1}^* X_m & Z_{m-1}^* X_1 & \cdots & Z_{m-1}^* X_{m-1} \end{bmatrix} \Big/ D_{m-1} \right)$$

$$= X_m - [X_1, \ldots, X_{m-1}] D_{m-1}^{-1} \begin{bmatrix} Z_1^* X_m \\ \vdots \\ Z_{m-1}^* X_m \end{bmatrix}.$$

So

$$Z_m^* G_{m-1,m} = Z_m^* X_m - [Z_m^* X_1, \ldots, Z_m^* X_{m-1}] D_{m-1}^{-1} \begin{bmatrix} Z_1^* X_m \\ \vdots \\ Z_{m-1}^* X_m \end{bmatrix}$$

$$= \left(\begin{bmatrix} & D_{m-1} & & \begin{matrix} Z_1^* X_m \\ \vdots \end{matrix} \\ Z_m^* X_1 & \cdots & Z_m^* X_{m-1} & Z_m^* X_m \end{bmatrix} \Big/ D_{m-1} \right)$$

$$= (D_m / D_{m-1}).$$

Applying (2.3), we get

$$|Z_m^* G_{m-1,m}| = \frac{|D_m|}{|D_{m-1}|}. \qquad \blacksquare$$

The following theorem is an immediate consequence of Lemma 4.1.

THEOREM 4.1. *The MRPA and the MRIA are well defined if and only if D_k is a block strongly nonsingular matrix.*

Proof. The MRPA and the MRIA are well defined if and only if, for $m = 1, \ldots, k$, the matrices $Z_m^* G_{m-1,m}$ are nonsingular, so if and only if

$|Z_m^* G_{m-1,m}| = |D_m|/|D_{m-1}| \neq 0$; but $|D_0| = 1$, and thus $|Z_m^* G_{m-1,m}| \neq 0$ if and only if $|D_m| \neq 0$. ■

LEMMA 4.2. *For $m = 1, \dots, k$ and $i > j \geqslant m$, we have $Z_m^* G_{j,i} = 0$.*

Proof. We proceed by induction. For $j = m$ and $i > j$ we have

$$\begin{aligned} Z_m^* G_{m,i} &= Z_m^*\left(G_{m-1,i} - G_{m-1,m}\left(Z_m^* G_{m-1,m}\right)^{-1} Z_m^* G_{m-1,i}\right) \\ &= Z_m^* G_{m-1,i} - Z_m^* G_{m-1,m}\left(Z_m^* G_{m-1,m}\right)^{-1} Z_m^* G_{m-1,i} \\ &= 0. \end{aligned}$$

Assume that the theorem is true for $j > m$; we shall prove it for $j + 1$. We have for $i > j + 1$

$$G_{j+1,i} = G_{j,i} - G_{j,j+1}\left(Z_{j+1}^* G_{j,j+1}\right)^{-1} Z_{j+1}^* G_{j,i},$$

$$Z_m^* G_{j+1,i} = Z_m^* G_{j,i} - Z_m^* G_{j,j+1}\left(Z_{j+1}^* G_{j,j+1}\right)^{-1} Z_{j+1}^* G_{j,i}.$$

By the induction hypothesis, the lemma follows. ■

THEOREM 4.2. *For $j = 1, \dots, k$ and $i = j, \dots, k$, we have $Z_j^* E_i = 0$ and $Z_j^* P_i = U_j = Z_j^* Y$.*

Proof. For $i = j$ we have

$$\begin{aligned} Z_j^* E_j &= Z_j^*\left(E_{j-1} - G_{j-1,j}\left(Z_j^* G_{j-1,j}\right)^{-1} Z_j^* E_{j-1}\right) \\ &= Z_j^* E_{j-1} - Z_j^* G_{j-1,j}\left(Z_j^* G_{j-1,j}\right)^{-1} Z_j^* E_{j-1}) \\ &= Z_j^* E_{j-1} - Z_j^* E_{j-1} \\ &= 0, \end{aligned}$$

$$\begin{aligned} Z_j^* P_j &= Z_j^* P_{j-1} + Z_j^* G_{j-1,j}\left(Z_j^* G_{j-1,j}\right)^{-1} Z_j^*\left(Y - P_{j-1}\right) \\ &= Z_j^* P_{j-1} + Z_j^*\left(Y - P_{j-1}\right) \\ &= Z_j^* Y, \end{aligned}$$

and for $i > j$ we have

$$Z_j^* E_i = Z_j^* \left(E_j - \sum_{m=j}^{i-1} G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* E_m \right)$$

$$= Z_j^* E_j - \sum_{m=j}^{i-1} Z_j^* G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* E_m$$

$$= - \sum_{m=j}^{i-1} Z_j^* G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* E_m,$$

$$Z_j^* P_i = Z_j^* \left(P_j + \sum_{m=j}^{i-1} G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* (Y - P_m) \right)$$

$$= Z_j^* P_j + \sum_{m=j}^{i-1} Z_j^* G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* (Y - P_m)$$

$$= Z_j^* Y + \sum_{m=j}^{i-1} Z_j^* G_{m,m+1} \left(Z_{m+1}^* G_{m,m+1} \right)^{-1} Z_{m+1}^* (Y - P_m).$$

Applying Lemma 4.2, we get

$$Z_j^* E_i = 0,$$

$$Z_j^* P_i = Z_j^* P_j = Z_j^* Y. \qquad \blacksquare$$

We set, for $m = 1, \ldots, k$,

$$G_m = G_{m-1,m} \left(Z_m^* G_{m-1,m} \right)^{-1} Z_m^*,$$

$$Q_m = I - G_m,$$

$$S_m = Q_m Q_{m-1} \cdots Q_1.$$

It is easily verified that

$$G_m^2 = G_m, \qquad Q_m^2 = Q_m, \quad \text{and} \quad S_m^2 = S_m,$$

i.e., G_m, Q_m, and S_m are projectors.

REMARK 2. We have

$$\begin{aligned} E_m &= (I - G_m) E_{m-1} = Q_m E_{m-1}, \\ P_m &= (I - G_m) P_{m-1} + G_m Y = Q_m P_{m-1} + G_m Y, \\ G_{m,i} &= (I - G_m) G_{m-1,i} = Q_m G_{m-1,i}. \end{aligned}$$

It is easy to show the following result.

LEMMA 4.3. *Assume that D_k is a block strongly nonsingular matrix. We have*

(1) $G_i G_j = 0$ *for* $j > i$ (*i.e.* $G_1, \dots, G_k$ *are conjugate projectors*);
(2) $G_i G_{j-1,j} = 0$ *for* $j > i$;
(3) $G_i G_{i-1,i} = G_{i-1,i}$;
(4) $G_i S_j = 0$ *for* $j \geqslant i$;
(5) $S_i G_{j-1,j} = 0$ *for* $i \geqslant j$;
(6) $S_i G_{j-1,j} = G_{j-1,j}$ *for* $j > i$;
(7) $S_i^* Z_j = 0$ *for* $i \geqslant j$;
(8) $S_i S_j = S_j S_i = S_i$ *for* $i \geqslant j$.

If for $i = 1, \dots, k$ we set

$$\begin{aligned} \tilde{Z}_i &= [Z_1, \dots, Z_i], \\ \bar{Z}_i &= [Z_1, S_1^* Z_2, \dots, S_{i-1}^* Z_i], \\ W_i &= \left[G_{0,1}(Z_1^* G_{0,1})^{-1}, \dots, G_{i-1,i}(Z_i^* G_{i-1,i})^{-1}\right], \end{aligned}$$

we get the following result:

THEOREM 4.3. *Assume that D_k is a block strongly nonsingular matrix. We have*

(1) $S_i^* \tilde{Z}_j = 0$ *for* $i \geqslant j$;
(2) $\bar{Z}_i^* W_i = I$;
(3) $\tilde{Z}_i^* W_i$ *is a block lower triangular matrix with all diagonal entries equal to* 1;
(4) $\tilde{Z}_i = \tilde{Z}_i W_i \bar{Z}_i^*$.

Proof. (1): For $i \geqslant j$ we have from (7) of Lemma 4.3

$$\begin{aligned} S_i^* \tilde{Z}_j &= S_i^* [Z_1, \dots, Z_j] \\ &= [S_i^* Z_1, \dots, S_i^* Z_j] \\ &= 0. \end{aligned}$$

(2):

$$\bar{Z}_i^* W_i = \begin{bmatrix} Z_1^* \\ Z_2^* S_1 \\ \vdots \\ Z_i^* S_{i-1} \end{bmatrix} \left[G_{0,1}(Z_1^* G_{0,1})^{-1}, \ldots, G_{i-1,i}(Z_i^* G_{i-1,i})^{-1} \right]$$

$$= \left[(\bar{Z}_i^*)_l (W_i)_m \right]_{1 \leqslant l, m \leqslant i}$$

with

$$(\bar{Z}_i^*)_l (W_i)_m = Z_l^* S_{l-1} G_{m-1,m} (Z_m^* G_{m-1,m})^{-1} \quad \text{and} \quad S_0 = I.$$

For $l = m$ we have from (6) of Lemma 4.3

$$S_{m-1} G_{m-1,m} = G_{m-1,m}.$$

So we get

$$(\bar{Z}_i^*)_m (W_i)_m = I.$$

For $l > m$ we have from (5) of Lemma 4.3

$$S_{l-1} G_{m-1,m} = 0.$$

For $l < m$ we have from (6) of Lemma 4.3

$$S_{l-1} G_{m-1,m} = G_{m-1,m}.$$

So we get

$$Z_l^* S_{l-1} G_{m-1,m} = Z_l^* G_{m-1,m} = 0.$$

(3):

$$\tilde{Z}_i^* W_i = \begin{bmatrix} Z_1^* \\ \cdots \\ Z_i^* \end{bmatrix} \left[G_{0,1}(Z_1^* G_{0,1})^{-1}, \ldots, G_{i-1,i}(Z_i^* G_{i-1,i})^{-1} \right]$$

$$= \left[(\tilde{Z}_i^*)_l (W_i)_m \right]_{1 \leqslant l, m \leqslant i}$$

with

$$\left(\tilde{Z}_i^*\right)_l (W_i)_m = Z_l^* G_{m-1,m}\left(Z_m^* G_{m-1,m}\right)^{-1}.$$

For $l = m$ we have

$$Z_m^* G_{m-1,m}\left(Z_m^* G_{m-1,m}\right)^{-1} = I.$$

For $l < m$ we have from Lemma 4.2

$$Z_l^* G_{m-1,m} = 0.$$

(4): We have

$$W_i \bar{Z}_i^* = \left[G_{0,1}\left(Z_1^* G_{0,1}\right)^{-1}, \dots, G_{i-1,i}\left(Z_i^* G_{i-1,i}\right)^{-1}\right] \begin{bmatrix} Z_1^* \\ Z_2^* S_1 \\ \vdots \\ Z_i^* S_{i-1} \end{bmatrix}$$

$$\begin{aligned}
&= \sum_{m=1}^{i} G_{m-1,m}\left(Z_m^* G_{m-1,m}\right)^{-1} Z_m^* S_{m-1} \\
&= \sum_{m=1}^{i} G_m S_{m-1} \\
&= \sum_{m=1}^{i} (I - Q_m) S_{m-1} \\
&= \sum_{m=1}^{i} (S_{m-1} - S_m) \\
&= I - S_i.
\end{aligned}$$

So from (1) of this theorem we get

$$\begin{aligned}
\tilde{Z}_i^* W_i \bar{Z}_i^* &= \tilde{Z}_i^* (I - S_i) \\
&= \tilde{Z}_i^*. \qquad \blacksquare
\end{aligned}$$

REMARK 3. We have

$$E_k = S_k E_0,$$
$$G_{k,i} = S_k G_{0,i} = S_k X_i,$$
$$P_k = S_k P_0 + \sum_{j=1}^{k} R_{k,j} G_j Y$$

with

$$R_{k,j} = \prod_{i=1}^{k-j} Q_{k-i+1} = \prod_{i=j}^{k-1} Q_{k+j-1} = Q_k Q_{k-1} \cdots Q_{j+1}$$

and

$$R_{k,k} = I.$$

THEOREM 4.4. *If we set* $R_k = [R_{k,1}G_{0,1}(Z_1^* G_{0,1})^{-1}, \dots, R_{k,k}G_{k-1,k}(Z_k^* G_{k-1,k})^{-1}]$, *we have* $S_k + R_k \tilde{Z}_k^* = I$.

Proof. We have

$$R_k \tilde{Z}_k^* = \left[R_{k,1}G_{0,1}(Z_1^* G_{0,1})^{-1}, \dots, R_{k,k}G_{k-1,k}(Z_k^* G_{k-1,k})^{-1} \right] \begin{bmatrix} Z_1^* \\ \vdots \\ Z_k^* \end{bmatrix}$$

$$= \sum_{j=1}^{k} R_{k,j} G_{j-1,j} (Z_j^* G_{j-1,j})^{-1} Z_j^*$$
$$= \sum_{j=1}^{k} R_{k,j} G_j$$
$$= \sum_{j=1}^{k} R_{k,j} (I - Q_j)$$
$$= -Q_k Q_{k-1} \cdots Q_1 + R_{k,k}$$
$$= I - S_k. \quad \blacksquare$$

COROLLARY 4.1. *For $i = 1, \dots, k$, we have*

$$S_i = I - W_i \bar{Z}_i^*$$
$$= I - W_i\left(\tilde{Z}_i^* W_i\right)^{-1} \tilde{Z}_i^*$$
$$= I - R_i \tilde{Z}_i^*.$$

Proof. We have

$$\begin{aligned}
S_i &= I - W_i \bar{Z}_i^* && \text{from the proof of Theorem 4.3(4)}\\
&= I - W_i\left(\tilde{Z}_i^* W_i\right)^{-1} \tilde{Z}_i^* W_i \bar{Z}_i^* && \text{from Theorem 4.3(3)}\\
&= I - W_i\left(\tilde{Z}_i^* W_i\right)^{-1} \tilde{Z}_i^* && \text{from Theorem 4.3(4)}\\
&= I - R_i \tilde{Z}_i^* && \text{from Theorem 4.4.}
\end{aligned}$$ ■

COROLLARY 4.2. *If D_k is a block strongly nonsingular matrix and if $n_1 + n_2 + \cdots + n_k = n$, we have $S_k = 0$, $E_k = 0\ \forall E_0$, and $P_k = Y\ \forall P_0$.*

Proof. We have from Corollary 4.1

$$S_k = I - W_k\left(\tilde{Z}_k^* W_k\right)^{-1} \tilde{Z}_k^*.$$

So if $n_1 + n_2 + \cdots + n_k = n$ and D_k is a block strongly nonsingular matrix, then the matrices $\tilde{Z}_k^*$ and W_k are nonsingular and

$$S_k = I - W_k W_k^{-1} \tilde{Z}_k^{*-1} \tilde{Z}_k^* = 0.$$

From Remark 3 we have

$$E_k = S_k E_0 = 0 \qquad \forall E_0$$

and

$$
\begin{aligned}
P_k &= S_k P_0 + \sum_{j=1}^{k} R_{k,j} G_j Y \\
&= R_k \tilde{Z}_k^* Y \\
&= Y. \qquad \blacksquare
\end{aligned}
$$

Applications of the MRPA and the MRIA for solving a block or a matrix linear system and their connections with some other methods used in numerical analysis are under investigation.

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